## References

[1] Benthem, J. P., "On the analysis of swept wing structures," National Aero- and Astronautical Research Inst. Rept. S-578, Amsterdam (June 1961).

[2] Dill, E. H. and Ortega, M. A., "Derivation of a stiffness matrix for a rectangular plate element in bending," Boeing Co. SARM 13, Seattle (June 1960).

[3] Grafton, P. E. and Strome, D. R., "Analysis of axisymmetric shells by the direct stiffness method," Boeing Co. SARM 36, Seattle (January 11, 1962).

[4] Levy, S., "Structural analysis and influence coefficients for delta wings," J. Aeronaut. Sci. **20**, 449–454 (1953).

[5] Lunder, C. A., "Stiffness matrix for a triangular, orthotropic disk," M. S. Thesis, Univ. Washington, Seattle (1961).

[6] Martin, H. C., "Plane elasticity problems and the direct stiffness method," Trend in Eng. **13**, 5 (January 1961).

[7] Melosh, R. J., "A stiffness matrix for the analysis of thin plates in bending," J. Aerospace Sci. **28**, 34–43 (1961).

[8] Melosh, R. J. and Merritt, R. G., "Evaluation of spar matrices for stiffness analyses," J. Aerospace Sci. **25**, 537–543 (1958).

[9] Ortega, M. A., "Calculation of the deflection at the center of a square plate with clamped edges under a concentrated load at the center, by making use of different stiffness matrices," Boeing Co. SARM 14, Seattle (May 31, 1960).

[10] Ortega, M. A., "Calculation of the deflection at the center of a square plate with simply-supported edges under a concentrated load at the center, by making use of different stiffness matrices," Boeing Co. SARM 15, Seattle (May 31, 1960).

[11] Papenfuss, B. W., "Lateral plate deflection by stiffness matrix methods," M. S. Thesis, Univ. Washington, Seattle (1959).

[12] Prager, W., "The extremum principles of the mathematical theory of elasticity and their use in stress analysis," Univ. Washington Eng. Expt. Station Bull. 119, Seattle (1951).

[13] Renton, J. D., "Stability of space frames by computer analysis," J. Struct. Div. Am. Soc. Civil Engrs. **88**, 81–103 (August 1962).

[14] Steffensen, J. F., *Interpolation* (Williams and Wilkins Co., Baltimore, Md., 1927), Chap. IV.

[15] Synge, J. L., *The Hypercircle in Mathematical Physics* (Cambridge University Press, Cambridge, 1957), pp. 98–117, 209–213.

[16] Turner, M. J., "The direct stiffness method of structural analysis," AGARD Structures and Materials Panel Meeting, Aachen, Germany (September 17, 1959).

[17] Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J., "Stiffness and deflection analysis of complex structures," J. Aeronaut. Sci. **23**, 805–823 (1956).

[18] Turner, M. J., Dill, E. H., Martin, H. C., and Melosh, R. J., "Large deflections of structures subjected to heating and external loads," J. Aerospace Sci. **27**, 97–107 (1960).


---

JULY 1963 AIAA JOURNAL VOL. 1, NO. 7

# Interception of High-Speed Target by Beam Rider Missile


ODIN R. S. ELNAN*
*University of Cincinnati, Cincinnati, Ohio*

AND

HSU LO†
*Purdue University, Lafayette, Ind.*



**The trajectories of a line-of-sight beam rider missile are determined for the case where the target is moving at suborbital speeds and the missile speed is smaller by a factor of 2 or 3. Of primary interest are the conditions under which interception may take place: the minimum target engagement range and interception range as functions of missile-to-target speed ratio and missile normal acceleration capability. The results of this analysis show that interception will take place for an arbitrary speed ratio with a reasonable value of lateral acceleration if the range at which the missile engages the target can be made sufficiently large.**


## Nomenclature

$r,\theta$ = plane polar coordinates in reference frame of Fig. 1
$x,y$ = rectangular coordinates in reference frame of Fig. 1
$t$ = missile flight time
$V$ = speed
$R$ = crossover range defined in Fig. 1
$c$ = defined in Eq. (4)
$u$ = missile lead angle
$\gamma$ = missile flight path angle
$\tau$ = missile-to-target speed ratio
$a_n$ = lateral missile acceleration

*Superscripts*

$\cdot$ = differentiation with respect to time
$'$ = differentiation with respect to $\theta$
$*$ = intercept values at maximum value of $y_m$

*Subscripts*

$m$ = missile values
$t$ = target values
$L$ = values at missile launch
$I$ = values at interception

Received by ARS October 11, 1962; revision received April 5, 1963.

* Assistant Professor of Aerospace Engineering. Member AIAA.

† Professor of Aeronautical and Engineering Sciences. Member AIAA.

## I. Introduction

A MISSILE guided by the so-called line-of-sight beam rider method has been shown to be capable of intercepting conventional-type targets, such as manned aircraft at moderate range, if the missile has a speed advantage.

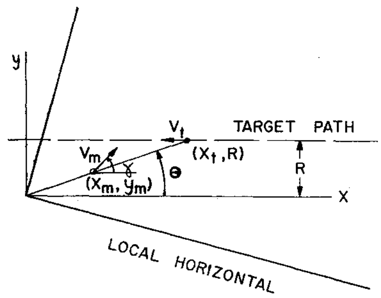


**Fig. 1 Geometry of missile and target motion**

The trajectories, flight times, and normal accelerations have been obtained by a series solution in Ref. 1. The problem of control and capture of such missiles is considered in Ref. 2.

The object of this paper is to ascertain the intercept capabilities of the beam rider missile against targets whose speeds may be on the order of 16,000 mph. For the sake of simplicity, a nonmaneuvering target moving in a straight line is considered. The missile is treated as a point mass moving at constant speed. A significant feature of this analysis is that the investigation is limited to the conditions under which the target has a speed advantage. The geometry of the intercept problem is shown in Fig. 1.

## II. Missile Trajectories

Since the missile always is located in the guidance beam that tracks the target, the differential equation of the flight path is given by

$$\dot{r}_m{}^2 + r^2\dot{\theta}^2 = V_m{}^2 \tag{1}$$

where $\dot{\theta}$ is prescribed by the target motion.

Since the target moves parallel to the $x$ axis with constant speed,

$$t = (R/V_t)(\cot\theta_L - \cot\theta) \tag{2}$$

such that

$$\dot{\theta} = (V_t/R)\sin^2\theta \tag{3}$$

and Eq. (1) becomes

$$r_m{}^2 + r_m{}'^2 = c^2 \csc^4\theta \tag{4}$$

where

$$c = RV_m/V_t$$

Introducing the lead angle $u$ and the flight path angle $\gamma$,

$$r_m = c \csc^2\theta \sin u \tag{5}$$

$$u = \gamma - \theta \tag{6}$$

Using Eqs. (5) and (6) in (4), one obtains

$$[d(\cot\theta)/d\gamma] + \tfrac{1}{2}\cot\theta\cot\gamma + \tfrac{1}{2} = 0 \tag{7}$$

which is a linear differential equation in $\cot\theta$. The fact that this differential equation governs the missile flight path al-

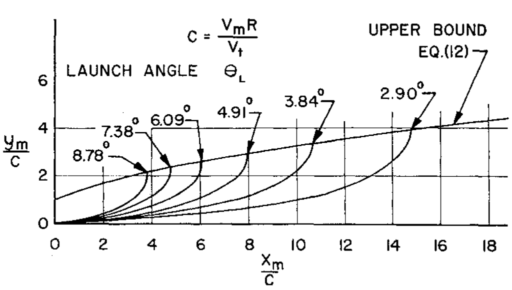


**Fig. 2 Generalized missile trajectories**

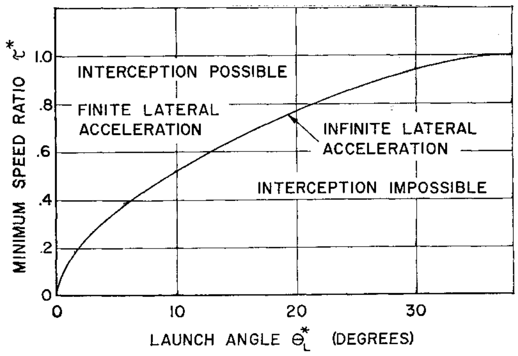


**Fig. 3 Minimum speed ratio required for interception**

ready has been noted in Ref. 3.

Integrating Eq. (7), using the initial conditions $\theta = \theta_L$ when $\gamma = \gamma_L$,

$$\cot\theta = (\sin\gamma)^{-1/2}\left[\cot\theta_L(\sin\gamma_L)^{1/2} - \frac{1}{2}\int_{\gamma_L}^{\gamma}(\sin\eta)^{1/2}d\eta\right] \tag{8}$$

The integral on the right-hand side may be transformed to elliptic integrals, such that the solution becomes

$$\cot\theta = (\sin\gamma)^{-1/2}\{\cot\theta_L(\sin\gamma_L)^{1/2} + \tfrac{1}{2}2^{1/2}[2E(\phi,k) - F(\phi,k) - 2E(\phi_L,k) + F(\phi_L,k)]\} \tag{9}$$

where $F$ and $E$ are normal elliptic integrals of first and second kind with moduli $k = \frac{1}{2}(2^{1/2})$ and arguments

$$\phi = \text{arc}\cos(\sin\gamma)^{1/2}$$

The missile coordinates then are given by

$$x_m = c\cos\theta\csc^2\theta\sin u \tag{10}$$

$$y_m = c\csc\theta\sin u \tag{11}$$

and the flight time is found from Eq. (2).

It is convenient to exhibit the missile trajectories in the generalized form of $y_m/c$ vs $x_m/c$ as shown in Fig. 2. Only those trajectories that will intercept for a speed ratio $\tau < 1$ are shown. It may be demonstrated that these trajectories correspond to a launch angle range $0 < \theta_L < 37°$. A comprehensive discussion of all beam rider trajectories is contained in Ref. 4.

The upper bound on the solutions to Eq. (4) is given by $u = \pi/2$ or $r_m = c\csc^2\theta$. In terms of the missile coordinates, this relation becomes

$$x_m/c = [(y_m/c)^4 - (y_m/c)^2]^{1/2} \tag{12}$$

The target path would appear in Fig. 2 as the curve $y_t/c = 1/\tau$. Hence, it is seen that, for a specified speed ratio, there are launch angles that will not yield interception. From

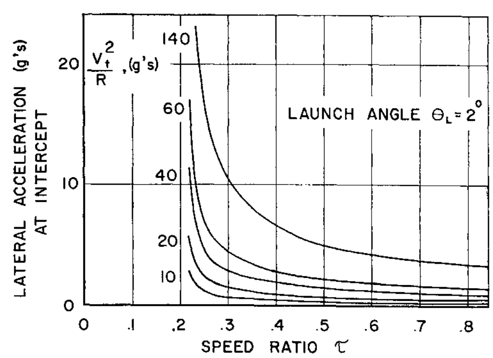


**Fig. 4 Effect of speed ratio on lateral acceleration at intercept**

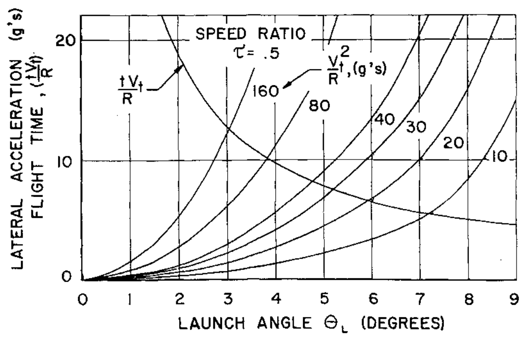


**Fig. 5 Effect of launch angle on lateral acceleration at intercept and flight time**

Eq. (11), the minimum value of speed ratio which gives interception at a particular launch angle is

$$\tau^* = \sin\theta^* \tag{13}$$

where $\theta^*$ is polar angle corresponding to intercept at $u = \pi/2$. The relation between $\tau^*$ and $\theta_L^*$ is shown in Fig. 3. Equations (12) and (13) may be used to provide the minimum value of intercept range:

$$r_{mI}^* = R/\tau^* \tag{14}$$

and engagement range:

$$r_{tL}^* = R/\sin\theta_L^* \tag{15}$$

for specified values of speed ratio.

## III. Lateral Acceleration Limit

A limiting factor on the use of beam rider missiles is the maximum lateral acceleration that the missile may attain. For the two-dimensional case considered here, the acceleration required to maintain the flight path is

$$a_n = V_m\dot{\gamma} \tag{16}$$

If Eqs. (7) and (3) are used,

$$a_n = \frac{2\tau V_t^2 \sin\theta \sin\gamma}{R \cos u} \tag{17}$$

For an incoming target and speed ratio $\tau < 1$, $a_n$ increases monotonically with $\theta$ and approaches $\infty$ as $u$ approaches $\pi/2$. The sensitivity of the lateral acceleration at interception to speed ratio changes at a given launch angle may be inferred from Fig. 4.

It is only in the neighborhood of the critical value $\tau^*$ that $a_n$ increases rapidly. This may be accounted for on the basis that the effects of missile speed $V_m$ and trajectory curvature on acceleration have a tendency to offset each other for values of $\tau$ larger than $\tau^*$.

**Table 1 Engagement range, intercept range, and flight time for various crossover ranges**

| $R$, miles | $r_{tL}$, miles | $r_{tL}^*$, miles | $r_{mI}$, miles | $r_{mI}^*$, miles | $t$, sec |
|---|---|---|---|---|---|
| 5.8 | 159 | 35 | 53 | 12 | 24 |
| 11.4 | 225 | 69 | 75 | 23 | 34 |
| 20.6 | 308 | 125 | 102 | 41 | 47 |
| 35.6 | 415 | 216 | 138 | 71 | 64 |
| 57.5 | 542 | 348 | 180 | 115 | 83 |

Figure 5 shows the variation of acceleration at intercept and nondimensional flight time $tV_t/R$ with launch angle for $\tau = 0.5$. It is evident that the crossover range and engagement range are important parameters.

This effect is illustrated in Table 1, where the values of engagement range $r_{tL}$, intercept range $r_{mI}$, and flight time have been computed for various crossover ranges and for target and missile speeds of 16,000 and 8000 mph, respectively. The allowable value of lateral acceleration is taken to be 20 $g$'s. The minimum values of $r_{tL}$ and $r_{mL}$ as found from Eqs. (14) and (15) likewise are shown.

It is seen from Table 1 that, if a missile-target engagement range of 540 miles is provided, all targets having impact points within a 58-mile-radius circle of the launch point may be intercepted. The maximum value of intercept range is 180 miles. For a larger value of the allowable lateral acceleration or increased missile speed, the values of $r_{tL}$ and $r_{mI}$ corresponding to the same values of $R$ would decrease.

The methods of analysis and the data provided herein also may be used for purposes of computing the decision time-to-launch available from the instant the target is detected.

## IV. Conclusion

The preceding analysis has served to illustrate the capabilities of a line-of-sight beam rider missile deployed against a high-speed target. The limiting features in such an application have been suggested, and the simplified analysis has served to indicate that, if missile launch is undertaken at sufficiently large target range, this particular method of navigation may be successful. A more exact analysis, including the effects of thrusting and aerodynamic drag on the missile trajectory, is a logical extension of this preliminary analysis.

## References

[1] Locke, A. S., *Guidance* (D. Van Nostrand Co. Inc., New York, 1955), Chap. 12, p. 446.

[2] McDonough, V. J., "Control of surface-to-air missiles," Sperry Eng. Rev. **14,** 2–10 (1961).

[3] Seifert, W. W. and Steeg, C. W., Jr., *Control Systems Engineering* (McGraw Hill Book Co. Inc., New York, 1960), p. 112.

[4] Wilder, C. E., "A discussion of a differential equation," Am. Math. Monthly **38,** 17–21 (1931).